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Pressure structure functions and spectra for locally isotropic turbulence

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Beginning with the known relationship between the pressure structure function and the fourth-order two-point correlation of velocity derivatives, we obtain a new theory relating the pressure structure function and spectrum to fourth-order velocity structure functions. This new theory is valid for all Reynolds numbers and for all spatial separations and wavenumbers. We do not use the joint Gaussian assumption that was used in previous theory. The only assumptions are local homogeneity, local isotropy, incompressibility, and use of the Navier–Stokes equation. Specific formulae are given for the mean-squared pressure gradient, the correlation of pressure gradients, the viscous range of the pressure structure function, and the pressure variance. Of course, pressure variance is a descriptor of the energy-containing range. Therefore, for any Reynolds number, the formula for pressure variance requires the more restrictive assumption of isotropy. For the case of large Reynolds numbers, formulae are given for the inertial range of the pressure structure function and spectrum and of the pressure-gradient correlation; these are valid on the basis of local isotropy, as are the formulae for mean-squared pressure gradient and the viscous range of the pressure structure function. Using the experimentally verified extension to fourth-order velocity structure functions of Kolmogorov's theory, we obtain $r^{4/3}$ and $k^{-7/3}$ laws for the inertial range of the pressure structure function and spectrum. The modifications of these power laws to account for the effects of turbulence intermittency are also given. New universal constants are defined; these require experimental evaluation. The pressure structure function is sensitive to slight departures from local isotropy, implying stringent conditions on experimental data, but applicability of the previous theory is likewise constrained. The results are also sensitive to compressibility.

1. Introduction

The pressure structure function is defined by

$$D_P(\boldsymbol{r}) \equiv \frac{1}{\rho^2}\langle (P-P')^2\rangle,$$

where P is pressure and ρ is density. Unprimed and primed quantities, e.g. P and P', are taken at spatial positions denoted by $\boldsymbol{x}$ and $\boldsymbol{x}'$, respectively; also, $\boldsymbol{r} \equiv \boldsymbol{x}-\boldsymbol{x}'$ and $r \equiv |\boldsymbol{r}|$. Angle brackets denote averaging. Obukhov (1949) related $D_P(\boldsymbol{r})$ to a two-point fourth-order velocity-derivative correlation. Independently, Batchelor (1951) obtained the pressure correlation in terms of this same derivative correlation. In addition, Batchelor (1951) related this derivative correlation, and hence the pressure correlation, to the fourth-order divergence of the fourth-order velocity correlation. To obtain tractable results, both of these authors used the joint Gaussian approximation for either velocities or velocity derivatives.

Without using the joint Gaussian approximation, Uberoi (1953) expressed Batchelor's (1951) formula for the pressure correlation in terms of integrals of linear combinations of components of the fourth-order velocity correlation, which is defined by

$$R_{ijkl}(\boldsymbol{r}) \equiv \langle u_i u_j u'_k u'_l \rangle, \tag{1}$$

where u_i, u_j, u'_k, and u'_l are velocity components. Clearly, $D_P(r)$ could also be expressed in terms of integrals of components of $R_{ijkl}(\boldsymbol{r})$.

That $D_P(r)$ should not be expressed in terms of $R_{ijkl}(\boldsymbol{r})$ (as distinct from its fourth-order divergence) is clear from several considerations. First, for small r and for very large Reynolds numbers, $D_P(r)$ depends only on the local behaviour of turbulence (Yaglom 1949), whereas $R_{ijkl}(\boldsymbol{r})$ depends on the energy-containing range. Secondly, there must be subtraction of very large values of $R_{ijkl}(\boldsymbol{r})$ to produce the relatively small quantities needed to obtain pressure correlations, spectra, and structure functions (Hill 1993). This implies that measurements of $R_{ijkl}(\boldsymbol{r})$ would have to be extremely precise to produce pressure quantities of modest accuracy. For example, Uberoi (1953) measured the linear combinations of components of $R_{ijkl}(\boldsymbol{r})$ for low-Reynolds-number turbulence, calculated the pressure correlation, and found that small differences between the relatively large quantities made the resulting pressure correlation extremely uncertain. For $D_P(r)$, this problem becomes increasingly severe as r is decreased within the inertial and dissipation ranges for very large Reynolds numbers.

Yaglom (1949) stated that for small r, the pressure structure function depends only on the local behaviour of turbulence. That is, pressure differences are local quantities. We are motivated by our conviction that statistics of products of pressure differences and of such products multiplied by velocity differences should be related to statistics of velocity differences, not to statistics of velocity products. Guided by this concept, we seek another fourth-order velocity statistic that is related to pressure fluctuations but does not contain large terms that cancel. A structure function consisting of the product of four differences of velocity components is the desired statistic. It is defined by

$$D_{ijkl}(\boldsymbol{r}) \equiv \langle (u_i - u'_i)(u_j - u'_j)(u_k - u'_k)(u_l - u'_l) \rangle. \tag{2}$$

This statistic has the simplest possible isotropic form, and one of its components has been extensively studied theoretically and experimentally. This allows us to derive a simple general formula relating the pressure structure function to integrals of components of this fourth-order velocity structure function, and to derive formulae for asymptotic ranges. Local isotropy and local scaling apply to both $D_P(r)$ and $D_{ijkl}(\boldsymbol{r})$; our derivation relates such scaling. We use the Navier–Stokes equation and incompressibility. To obtain the greatest range of applicability, we derive $D_P(r)$ on the basis of local homogeneity and local isotropy. These assumptions are less restrictive than assuming homogeneity and local isotropy or assuming isotropy.

We express our results in a manner that prepares them for experimental evaluation. This is accomplished by reducing the theory to asymptotic expressions that include universal constants and parameters that require experimental evaluation. Asymptotic results for the inertial range, which are given in §§6 and 7, are based on the empirically evaluated inertial-range formulae for $D_{ijkl}(\boldsymbol{r})$ as presented in §5. Asymptotic viscous-range results are also given in §§6 and 7.

Lengthy derivations are required for our results. These include derivation of a formula for the fourth-order divergence of $D_{ijkl}(\boldsymbol{r})$, as well as many integrations by parts. All these derivations were given in detail by Hill (1993).

2. The theory of 1948–1951

In the years 1948–1951, great progress was made in relating a variety of pressure statistics to velocity statistics and velocity-derivative statistics. Obukhov (1949) derived a differential equation for the pressure correlation, from which he obtained a differential equation for $D_P(r)$. Using this result, Yaglom (1949) derived the mean-squared pressure gradient. Heisenberg (1948) independently derived the mean-squared pressure gradient. Batchelor (1951) derived the pressure correlation from which he obtained the pressure structure function and spectrum, mean-squared pressure gradient, and pressure-gradient correlation. Obukhov & Yaglom (1951) derived differential equations for the pressure correlation and, subsequently, the pressure structure function; they solved their differential equations to obtain the pressure structure function and derived the mean-squared pressure gradient and pressure-gradient correlation.

To obtain quantitative results from their theories, Obukhov (1949) and Obukhov & Yaglom (1951) assumed that velocity derivatives at two spatial points have the joint Gaussian probability distribution, whereas Batchelor (1951) assumed that velocities at two points are joint Gaussian, and Heisenberg (1948) assumed the statistical independence of Fourier components of velocities. Batchelor (1951) showed that Heisenberg's assumption produced the same results as the joint Gaussian assumption.

On the basis of the joint Gaussian assumption, Batchelor (1951) and Obukhov (1949) showed that the pressure structure function varies as $r^{4/3}$ within the inertial range; hence, the pressure spectrum varies as $k^{-7/3}$. Because of the use of the joint Gaussian assumption, the accuracy of this power-law exponent is in doubt, as are the value of the proportionality constant of the inertial-range power law, the mean-squared pressure gradient, the transition between the inertial and viscous subranges, the effects of turbulence intermittency, and the general formula relating the pressure structure function to velocity statistics. Without the joint Gaussian assumption, Obukhov & Yaglom (1951) obtained that $D_P(r) \propto \epsilon^{4/3} r^{4/3}$ on the basis of dimensional analysis using r and energy dissipation rate ϵ as parameters. Validity of this dimensional analysis needs confirmation.

We next discuss the derivations by Obukhov, Yaglom, and Batchelor with emphasis on the assumptions they made regarding local homogeneity versus homogeneity and local isotropy versus isotropy. The discussion follows the theory prior to introduction of the joint Gaussian assumption. As the first step, the divergence of the Navier–Stokes equation gives the following relationship of the Laplacian of pressure to the velocity derivatives for incompressible fluid (cf. Batchelor 1951):

$$\frac{1}{\rho}\partial_{ii} P = -\partial_{ij}(u_i u_j) = -\partial_j u_i \partial_i u_j. \tag{3}$$

Summation is implied over repeated Roman indices. Similar to Obukhov's (1949) notation, ∂_i denotes differentiation with respect to the coordinate x_i, and ∂_{ij} denotes differentiation with respect to both coordinates x_i and x_j with $\boldsymbol{x}'$ held fixed, and ∂'_{kl} denotes differentiation with respect to x'_k and x'_l with $\boldsymbol{x}$ held fixed. Thus, ∂_{ii} in (3) is the Laplacian operator; the rightmost expression in (3) is equivalent to the middle expression on the basis of incompressibility. Using (3) and homogeneity (but not isotropy), Obukhov (1949) and Obukhov & Yaglom (1951) obtained that

$$D_P(r)_{|iikk} = -2Q(r), \tag{4a}$$

where the derivative moment $Q(\boldsymbol{r})$ is

$$Q(\boldsymbol{r}) \equiv \langle \partial_{ij}(u_i u_j) \partial'_{kl}(u'_k u'_l) \rangle = \langle \partial_j u_i \partial_i u_j \partial'_l u'_k \partial'_k u'_l \rangle. \tag{4b}$$

A subscript vertical bar followed by indices, as in (4a), denotes differentiation with respect to components of $\boldsymbol{r}$; thus, the Laplacian with respect to $\boldsymbol{r}$ operates twice on the left-hand side of (4a). Batchelor (1951) used homogeneity and (3) to derive the pressure correlation; given homogeneity, as opposed to local homogeneity, his result is equivalent to (4a). Batchelor (1951) also defined $Q(\boldsymbol{r})$ as in (4b), and subsequently used homogeneity to express $Q(\boldsymbol{r})$ as the fourth-order divergence,

$$Q(\boldsymbol{r}) = R_{ijkl}(\boldsymbol{r})_{|ijkl}. \tag{4c}$$

Obukhov & Yaglom (1951) mentioned that (4a, b) are valid on the basis of local homogeneity; in the Appendix, we prove that this is correct.

On the basis of isotropy, Obukhov & Yaglom (1951) solved (4a) for $D_P(r)$, and Batchelor (1951) obtained an equivalent result for the pressure correlation. They obtained

$$D_P(r) = \frac{1}{3r}\int_0^r (y^4 - 3ry^3 + 3r^2y^2)\, Q(y)\,\mathrm{d}y + \frac{r^2}{3}\int_r^\infty yQ(y)\,\mathrm{d}y. \tag{5}$$

The integration variable y is spatial separation; throughout this paper, y is used instead of r when r appears in the limits of integration. The fact that Obukhov & Yaglom (1951) derived (5) after $Q(r)$ is simplified using the joint Gaussian assumption does not detract from the generality of their result. It is clear from the content of Yaglom's (1949) paper that he had obtained (5). As mentioned by Obukhov & Yaglom (1951), (5) follows from (4a, b) on the less restrictive basis of local isotropy. This is immediately clear from (4a, b) because both $D_P(r)$ and $Q(r)$ depend only on the local behaviour of turbulence. In the following, we use (5) to obtain new results.

3. General formula for the pressure structure function and spectrum

Using the distributive law of multiplication on the product of velocity differences in (2), we obtain

$$D_{ijkl}(\boldsymbol{r}) = -S_{ijkl}(\boldsymbol{r}) - S_{ikjl}(\boldsymbol{r}) - S_{iljk}(\boldsymbol{r}) + M_{ijkl}(\boldsymbol{r}), \tag{6a}$$

where

$$S_{ijkl}(\boldsymbol{r}) \equiv \langle (u_i u_j - u'_i u'_j)(u_k u_l - u'_k u'_l) \rangle \tag{6b}$$

$$= \langle u_i u_j u_k u_l \rangle + \langle u'_i u'_j u'_k u'_l \rangle - R_{ijkl}(\boldsymbol{r}) - R_{klij}(\boldsymbol{r}), \tag{6c}$$

$$M_{ijkl}(\boldsymbol{r}) \equiv B_{ijkl}(\boldsymbol{r}) + B_{jikl}(\boldsymbol{r}) + B_{kijl}(\boldsymbol{r}) + B_{lijk}(\boldsymbol{r}), \tag{6d}$$

and

$$B_{ijkl}(\boldsymbol{r}) \equiv \langle (u_i - u'_i) u_j u_k u_l \rangle + (\langle u'_i - u_i) u'_j u'_k u'_l \rangle. \tag{6e}$$

Note that (6a) is the average of an algebraic identity; likewise for (6b, c) above. On the basis of local homogeneity and incompressibility, we have (see Appendix)

$$B_{ijkl}(\boldsymbol{r})_{|i} = 0, \tag{7}$$

and, therefore, from (6d),

$$M_{ijkl}(\boldsymbol{r})_{|ijkl} = 0. \tag{8}$$

Performing the fourth-order divergence of (6c), and assuming homogeneity and substituting (4c), we have

$$S_{ijkl}(\boldsymbol{r})_{|ijkl} = -2Q(\boldsymbol{r}). \tag{9}$$

In the Appendix, we show the more fundamental result that (9) is valid on the basis of local homogeneity with $Q(\boldsymbol{r})$, as defined in (4*b*). Performing the fourth-order divergence of (6*a*), and using (8) and (9), gives

$$Q(\boldsymbol{r}) = \tfrac{1}{6} D_{ijkl}(\boldsymbol{r})_{|ijkl}, \tag{10}$$

which is valid on the basis of local homogeneity (see Appendix; isotropy is not needed). Comparing (10) and (4*c*) shows that $D_{ijkl}(\boldsymbol{r})_{|ijkl}$ is six times greater than $R_{ijkl}(\boldsymbol{r})_{|ijkl}$, despite the fact that (6*a*) expresses $D_{ijkl}(\boldsymbol{r})$ as the difference of quantities much larger than itself (Hill 1993). Thus, compared to $R_{ijkl}(\boldsymbol{r})$, $D_{ijkl}(\boldsymbol{r})$ does not have the same degree of cancellation of large terms that produce the small value of $D_P(\boldsymbol{r})$.

Henceforth, we need not repeat which of our results require local homogeneity or local isotropy because the required assumptions are obvious from our notation. If the vector separation $\boldsymbol{r}$ appears as the argument on the right-hand side of an equation, then local homogeneity is required, but local isotropy is not required; examples are (4*a*), (9), and (10). If spacing r, wavenumber k, or wavenumber component k_1 appears as an argument on the right-hand side of an equation, then local isotropy is required, as is local homogeneity (in contrast to homogeneity); an example is (5). If an equation violates these rules, then we explicitly state the required assumption; an example is our statement regarding the requirement of homogeneity for (4*c*). No assumption is required for definitions, so definitions are distinguished by using identity ($\equiv$) rather than equality ($=$); examples are (1), (2), (4*b*), and (6*b*, *d*, *e*).

We need a special coordinate system, which we call the preferred coordinate system. The preferred coordinate system is Cartesian with its 1-axis aligned along the separation vector $\boldsymbol{r}$. When we refer to specific components of the tensors, such as $D_{1111}(r)$ and $D_{2233}(r)$, we imply that these components are taken along axes of the preferred coordinate system. Thus, we will not repeat mention of the preferred coordinate system when we present results or refer to a tensor's components. These components depend only on the spacing r, not on all components of $\boldsymbol{r}$ separately. Greek indices are used to denote a general index for a component resolved in the preferred coordinate system [e.g. $D_{\alpha\alpha\beta\beta}(r)$]. No summation is implied by repeated Greek indices.

The tensor $D_{ijkl}(\boldsymbol{r})$ is symmetric under interchange of any pair of indices. Therefore, assuming local isotropy, its non-zero components in the preferred coordinate system are of the form $D_{\alpha\alpha\beta\beta}(r) = \langle (u_\alpha - u'_\alpha)^2 \, (u_\beta - u'_\beta)^2 \rangle$, where α and β may be 1, 2 or 3 (see Monin & Yaglom 1975, §13.3). $D_{ijkl}(\boldsymbol{r})$ can be specified by only three functions that are linear combinations of these non-zero components. We choose the functions $D_{1111}(r)$, $D_{\lambda\lambda\lambda\lambda}(r)$, and $D_{11\gamma\gamma}(r)$, where λ and γ are 2 or 3; γ can be equal to λ or different from λ. We note that local isotropy requires

$$3D_{2233}(r) = D_{\lambda\lambda\lambda\lambda}(r), \tag{11}$$

so $3D_{2233}(r)$ can replace $D_{\lambda\lambda\lambda\lambda}(r)$ in our formulae.

Hill (1993) obtained that the fourth-order divergence of $D_{ijkl}(\boldsymbol{r})$ is

$$\begin{aligned} D_{ijkl}(\boldsymbol{r})_{|ijkl} = {} & D^{(4)}_{1111}(r) + \frac{8}{r} D^{(3)}_{1111}(r) + \frac{12}{r^2} D^{(2)}_{1111}(r) \\ & - \frac{12}{r} D^{(3)}_{11\gamma\gamma}(r) - \frac{60}{r^2} D^{(2)}_{11\gamma\gamma}(r) - \frac{24}{r^3} D^{(1)}_{11\gamma\gamma}(r) + \frac{24}{r^4} D_{11\gamma\gamma}(r) \\ & + \frac{8}{r^2} D^{(2)}_{\lambda\lambda\lambda\lambda}(r) + \frac{8}{r^3} D^{(1)}_{\lambda\lambda\lambda\lambda}(r) - \frac{8}{r^4} D_{\lambda\lambda\lambda\lambda}(r). \end{aligned} \tag{12}$$

The superscripts in parentheses give the order of differentiation with respect to r.

Our general formulation for the pressure structure function is obtained by substituting (12) and (10) and the resulting $Q(r)$ into (5). Integration by parts reduces the general formulation to (Hill 1993)

$$D_P(r) = -\tfrac{1}{3}D_{1111}(r)+\tfrac{4}{3}r^2\int_r^{\infty} y^{-3}[D_{1111}(y)+D_{\lambda\lambda\lambda\lambda}(y)-6D_{11\gamma\gamma}(y)]\,\mathrm{d}y + \frac{4}{3}\int_0^r y^{-1}[D_{\lambda\lambda\lambda\lambda}(y)-3D_{11\gamma\gamma}(y)]\,\mathrm{d}y. \quad (13)$$

This result (13) is valid for all Reynolds numbers. Turbulence must be isotropic for application of (13) to the pressure variance and the energy-containing range. For locally isotropic turbulence, (13) can be used to obtain formulae for the inertial range, mean-squared pressure gradient, spectral representations, viscous range, and pressure-gradient correlations. A form of (13) that is more convenient for obtaining the last two quantities is

$$D_P(r) = \frac{\chi}{3}r^2-\tfrac{1}{3}D_{1111}(r)-\tfrac{4}{3}r^2\int_0^{r} y^{-3}[D_{1111}(y)+D_{\lambda\lambda\lambda\lambda}(y)-6D_{11\gamma\gamma}(y)]\,\mathrm{d}y + \frac{4}{3}\int_0^r y^{-1}[D_{\lambda\lambda\lambda\lambda}(y)-3D_{11\gamma\gamma}(y)]\,\mathrm{d}y, \quad (14)$$

where χ is the mean-squared pressure gradient that is derived in §4. Applying the joint Gaussian assumption to (13) and (14) and comparing this with previous results that use the joint Gaussian assumption verifies (13) and (14) (see Hill 1993, 1994). The relationship between a structure function and its spectrum from data along a line can be written as (Tatarskii 1971)

$$\Psi_P(k_1) = \frac{1}{\pi k_1}\int_0^{\infty}\mathrm{d}r\sin(k_1 r)\,D_P^{(1)}(r), \quad (15)$$

where $D_P^{(1)}(r) = \mathrm{d}D_P/\mathrm{d}r$, k_1 is the wave-vector component along the 1-axis, and $\Psi_P(k_1)$ is normalized so that the pressure variance equals the integral of $\Psi_P(k_1)$ from $k_1 = 0$ to ∞. Inserting (13) in (15) and integrating by parts gives

$$\Psi_P(k_1) = \frac{1}{\pi}\int_0^{\infty}\mathrm{d}r\frac{\sin(k_1 r)}{(k_1 r)}N_D(r)+\frac{8}{3\pi}\int_0^{\infty}\mathrm{d}r\left[\frac{\sin(k_1 r)}{(k_1 r)^3}-\frac{\cos(k_1 r)}{(k_1 r)^2}\right]A_D(r), \quad (16)$$

where

$$N_D(r) \equiv -\frac{r}{3}D_{1111}^{(1)}(r)-\tfrac{4}{3}D_{1111}(r)+4D_{11\gamma\gamma}(r),$$

and

$$A_D(r) \equiv D_{1111}(r)+D_{\lambda\lambda\lambda\lambda}(r)-6D_{11\gamma\gamma}(r).$$

The three-dimensional wavenumber spectrum is easily obtained from (16) because the three-dimensional spectrum equals $-k\Psi_P^{(1)}(k)$, which is the usual formula relating three- and one-dimensional isotropic scalar spectra; k denotes the magnitude of the wave vector. Note that an inertial-range formula is not to be substituted into (15) or (16); convergence of the integrals requires replacing (15) with the transform given in §6.

4. General formulation for the pressure variance, mean-squared pressure gradient, and pressure-gradient correlation

One can obtain pressure variance σ_P^2 by taking $r \to \infty$ in (13). Alternatively, we use the expression for pressure variance from Batchelor's (1951) equation (2.16):

$$\sigma_P^2 \equiv \frac{1}{\rho^2}\langle P^2 \rangle = -\frac{1}{2}\int_0^\infty \mathrm{d}r\, r^3 Q(r). \tag{17}$$

Using (10) and (12) in (17) and integrating by parts, we obtain

$$\sigma_P^2 \equiv -\tfrac{1}{6}D_{1111}(\infty) + \frac{2}{3}\int_0^\infty \mathrm{d}r\, r^{-1}[D_{\lambda\lambda\lambda\lambda}(r) - 3D_{11\gamma\gamma}(r)]. \tag{18}$$

In §5, we demonstrate convergence of the integral in (18). Of course, (17) and (18) require isotropy and are not valid for local isotropy.

We next consider the mean-squared pressure gradient χ defined by

$$\chi \equiv \frac{1}{\rho^2}\langle |\partial_i P|^2 \rangle.$$

χ is important in a number of practical problems, including particle dispersion, droplet growth, aerosol coagulation, and sound radiated by bubbles. We obtain the relationship of χ to $D_{ijkl}(\boldsymbol{r})$.

Expanding the pressure in a Taylor series, the definition of $D_P(r)$ gives $D_P(r) = (\chi/3)\, r^2$ as $r \to 0$. Comparing this with the limit $r \to 0$ applied to (13) gives

$$\chi = 4\int_0^\infty r^{-3}[D_{1111}(r) + D_{\lambda\lambda\lambda\lambda}(r) - 6D_{11\gamma\gamma}(r)]\,\mathrm{d}r. \tag{19}$$

Alternatively, Yaglom (1949) and Batchelor (1951) gave the relationship

$$\chi = \int_0^\infty rQ(r)\,\mathrm{d}r, \tag{20}$$

which can be obtained from (5). Substitution of (10) and (12) in (20) and integration by parts also yields (19). This serves to validate (19). Substitution of (19) in (13) gives (14).

Yaglom (1949), Batchelor (1951), and Obukhov & Yaglom (1951) showed that the correlation tensor of the acceleration vector consists of two terms: one is the pressure-gradient correlation that describes acceleration by the pressure gradient, and the other is acceleration by viscous friction. Batchelor (1951) and Monin & Yaglom (1975) showed that the former is much larger than the latter for very large Reynolds numbers, but the latter is the greater for very low Reynolds numbers. Satisfactory expressions for the viscous acceleration term were given by Obukhov & Yaglom (1951) and Monin & Yaglom (1975). Here, we give new results for the pressure-gradient correlation tensor defined by

$$A_{ij}(\boldsymbol{r}) \equiv \frac{1}{\rho^2}\langle \partial_i P\, \partial_j' P' \rangle \tag{21a}$$

$$= \tfrac{1}{2}D_P(\boldsymbol{r})_{|ij}. \tag{21b}$$

To obtain (21*b*) from (21*a*) on the basis of local homogeneity, we use the fact that $\partial_i \partial_j'(P-P')^2 = \partial_i \partial_j'(P^2 - 2PP' + P'^2) = -2\,\partial_i P\, \partial_j' P'$.

Given local isotropy, the usual formula for an isotropic second-rank tensor, e.g. equation (12.29) by Monin & Yaglom (1975), applies to $A_{ij}(\mathbf{r})$. Thus, in our preferred coordinate system, $A_{22}(r) = A_{33}(r)$ and $A_{\alpha\beta}(r) = 0$ if $\alpha \neq \beta$. Two components, $A_{11}(r)$ and $A_{\lambda\lambda}(r)$, where λ is 2 or 3, are sufficient to determine $A_{ij}(\mathbf{r})$. In addition, the curl of the gradient is identically zero, so $A_{11}(r)$ and $A_{\lambda\lambda}(r)$ are related by equation (12.70) of Monin & Yaglom (1975); therefore, $A_{ij}(\mathbf{r})$ is completely determined by either $A_{11}(r)$ or $A_{\lambda\lambda}(r)$. Assuming local isotropy, the second-order derivative of $D_P(\mathbf{r})$ in (21 b) gives

$$A_{\lambda\lambda}(r) = \frac{1}{2r} D_P^{(1)}(r), \qquad A_{11}(r) = \tfrac{1}{2} D_P^{(2)}(r). \tag{22a, b}$$

Substituting (14) into (22 a, b) gives the general expressions

$$A_{\lambda\lambda}(r) = \frac{\chi}{3} - \frac{1}{6r} D_{1111}^{(1)}(r) - \frac{2}{3r^2}[D_{1111}(r) - 3D_{11\gamma\gamma}(r)] - \frac{4}{3}\int_0^r y^{-3}[D_{1111}(r) + D_{\lambda\lambda\lambda\lambda}(r) - 6D_{11\gamma\gamma}(r)]\,\mathrm{d}y, \tag{23}$$

$$A_{11}(r) = \frac{\chi}{3} - \tfrac{1}{6} D_{1111}^{(2)}(r) - \frac{2}{3r}[D_{1111}(r) - 3D_{11\gamma\gamma}(r)]^{(1)} - \frac{2}{3r^2}[D_{1111}(r) + 2D_{\lambda\lambda\lambda\lambda}(r) - 9D_{11\gamma\gamma}(r)] - \frac{4}{3}\int_0^r y^{-3}[D_{1111}(y) + D_{\lambda\lambda\lambda\lambda}(y) - 6D_{11\gamma\gamma}(y)]\,\mathrm{d}y. \tag{24}$$

5. The r-dependence of the fourth-order velocity structure function

In this section, we establish some properties of $D_{ijkl}(\mathbf{r})$ that are needed for reduction of the general formulae in §§3 and 4 to specific formulae for asymptotic ranges. We first briefly discuss the r-dependence of this fourth-order structure function at very large (energy-containing range) and very small (dissipation range) separations; then we discuss the inertial-range r-dependence in detail.

For large separations, the following is readily obtained from (6 a–e):

$$D_{ijkl}(\infty) = 2[R_{ijkl}(0) + \sigma_{ij}\sigma_{kl} + \sigma_{ik}\sigma_{jl} + \sigma_{il}\sigma_{jk}], \tag{25}$$

where

$$\sigma_{ij} = \langle u_i u_j \rangle \tag{26}$$

is the single-point velocity covariance tensor. To obtain (25) from (6 a–e), we assumed homogeneity, statistical independence of u_i and u_j' for the case in which the distance between $\mathbf{x}$ and $\mathbf{x}'$ becomes infinite, and that $\langle u_i \rangle = 0$. For the moment, let no two of the indices α, γ or λ be equal; these indices may be 1, 2 or 3. From isotropy and (25), it can be shown that

$$D_{\alpha\alpha\alpha\alpha}(\infty) = D_{\lambda\lambda\lambda\lambda}(\infty) = 3D_{\lambda\lambda\gamma\gamma}(\infty) = 3D_{\alpha\alpha\gamma\gamma}(\infty), \tag{27}$$

which is not a repetition of (11) because (27) includes, for example, $D_{1111}(\infty) = 3D_{1133}(\infty) = 3D_{2233}(\infty)$. From (27), we see that the factor $[D_{\lambda\lambda\lambda\lambda}(r) - 3D_{11\gamma\gamma}(r)]$ in the integrand of (18) tends to zero as $r \to \infty$; therefore, the integral in (18) converges. Also, (27) shows that the first term in (18) can be written in a variety of ways.

Using a Taylor series expansion of $D_{\alpha\alpha\beta\beta}(r)$, we have the leading-order viscous-range formula

$$D_{\alpha\alpha\beta\beta}(r) = d_{\alpha\beta}\, r^4, \tag{28}$$

where

$$d_{\alpha\beta} \equiv \langle (\partial_1 u_\alpha)^2 (\partial_1 u_\beta)^2 \rangle. \tag{29}$$

The derivative moment (29) is a component of an eighth-order tensor. The notation $d_{\alpha\beta}$ is not meant to imply a second-rank tensor. Only italic subscripts denote a tensor. For future reference, we define

$$K_{\alpha\beta} \equiv d_{\alpha\beta}/d_1^2, \tag{30}$$

where

$$d_1 \equiv \langle (\partial_1 u_1)^2 \rangle.$$

Wyngaard & Tennekes (1970) named K_{11} the derivative kurtosis and showed that K_{11} varies with the Reynolds number.

We now turn to the inertial range of $D_{ijkl}(\boldsymbol{r})$. The velocity structure functions given by $\langle (u_1 - u_1')^n \rangle$ have been studied extensively in connection with the effects of intermittency. The relevance to the pressure structure function arises from $D_{1111}(r) \equiv \langle (u_1 - u_1')^4 \rangle$. The earliest theory of intermittency was given by Kolmogorov (1962). Some experimental studies of $D_{1111}(r)$ were by Stolovitzky, Sreenivasan & Juneja (1993), Anselmet *et al.* (1984), Antonia, Satyaprakash & Chambers (1982), Van Atta & Park (1972), and Van Atta & Chen (1970). The various theories of the intermittency effect are negligibly different when applied to $D_{1111}(r)$. These theories were reviewed by Anselmet *et al.* (1984), and progress continues (She & Leveque 1994). Here we state that for the inertial range,

$$D_{1111}(r) = C_{11}\, \epsilon^{4/3} r^q, \quad q \equiv 4/3 - 2\mu/9, \tag{31a, b}$$

where C_{11} depends on the flow macrostructure and ϵ is the energy dissipation rate per unit mass of fluid. Indeed, C_{11} has dimensions of a length raised to the power of $2\mu/9$. In (31 *b*), we express the exponent q in terms of the intermittency parameter μ, which is common in many studies. Sreenivasan & Kailasnath (1993) reviewed measurements of μ and estimated that $\mu = 0.25 \pm 0.05$, to which one may add, among others, a recent measurement $\mu = 0.20$ by Praskovsky & Oncley (1994) and a recent theoretical value $\mu = 2/9$ by She & Leveque (1994). For our purposes, it is significant that $2\mu/9 = 0.06$, which gives a very small departure from the 4/3 power law.

Although the empirical basis of (31 *a, b*) is for the case $\alpha = \beta = 1$, the dimensional analysis and subsequent averaging over local values of dissipation rate that produce (31 *a, b*) are equally valid for all of the non-zero components of $D_{ijkl}(\boldsymbol{r})$. Therefore, we have for the inertial range,

$$D_{\alpha\alpha\beta\beta}(r) = C_{\alpha\beta}\, \epsilon^{4/3} r^q. \tag{32}$$

We define

$$H_{\alpha\beta} \equiv C_{\alpha\beta}/C_{11}. \tag{33}$$

For instance, for r in the inertial range, we have

$$\begin{aligned} H_{22} &= D_{2222}(r)/D_{1111}(r) = D_{3333}(r)/D_{1111}(r) \\ &= 3D_{2233}(r)/D_{1111}(r) = H_{33} = 3H_{23} = 3H_{32}, \\ H_{12} &= D_{1122}(r)/D_{1111}(r) = D_{1133}(r)/D_{1111}(r) \\ &= H_{21} = H_{13} = H_{31}. \end{aligned}$$

Although the $C_{\alpha\beta}$ have macrostructure dependence, they very likely all have similar macrostructure dependence such that the $H_{\alpha\beta}$ are universal constants.

Inner scales $\ell_{\alpha\beta}$ are defined by equating viscous- and inertial-range formulae (28) and (32) at $r = \ell_{\alpha\beta}$, so that

$$\ell_{\alpha\beta} \equiv (C_{\alpha\beta}\,\epsilon^{4/3}/d_{\alpha\beta})^{1/(4-q)}. \tag{34}$$

Consider the dimensionless structure function components,

$$\tilde{D}_{\alpha\alpha\beta\beta}(r_{\alpha\beta}) \equiv D_{\alpha\alpha\beta\beta}(r)/C_{\alpha\beta}\,\epsilon^{4/3}\,\ell^{q}_{\alpha\beta}, \tag{35}$$

as functions of $r_{\alpha\beta} \equiv r/\ell_{\alpha\beta}$. In their viscous ranges and in their inertial ranges, the dimensionless structure functions (35) are equal to $r^4_{\alpha\beta}$ and $r^q_{\alpha\beta}$, respectively, for all α and β, and in these ranges they are independent of the Reynolds number; they differ in their transition between these ranges. The shape of this transition might depend on the Reynolds number. Use of Kolmogorov scaling, that is using the Kolmogorov microscale in place of $\ell_{\alpha\beta}$, produces scaled viscous-range formulae proportional to $K_{\alpha\beta}$. The $K_{\alpha\beta}$ are not necessarily equal for differing α and β, and the $K_{\alpha\beta}$ depend on the Reynolds number.

The ratio of derivative moments, $d_{\alpha\beta}$, is defined by

$$\Lambda_{\alpha\beta} \equiv d_{\alpha\beta}/d_{11}. \tag{36}$$

The $\Lambda_{\alpha\beta}$ are universal constants at high Reynolds numbers. The ratios of inner scales $\ell_{\alpha\beta}$ are also universal constants:

$$\ell_{\alpha\beta}/\ell_{11} = (H_{\alpha\beta}/\Lambda_{\alpha\beta})^{1/(4-q)}.$$

For use in the next section, we define the universal constants $h_{\alpha\beta}$ by

$$h_{\alpha\beta} \equiv \Lambda^{(2-q)/(4-q)}_{\alpha\beta}\,H^{2/(4-q)}_{\alpha\beta} = \Lambda_{\alpha\beta}(\ell_{\alpha\beta}/\ell_{11})^2 = H_{\alpha\beta}(\ell_{\alpha\beta}/\ell_{11})^{q-2}. \tag{37}$$

Henceforth, we will not state which of our equations require the phenomenological inertial-range equations (31 *a*, *b*) and (32) because this requirement is obvious from the appearance in an equation of one or more of the parameters ϵ, q, μ, $C_{\alpha\beta}$, $H_{\alpha\beta}$, $\ell_{\alpha\beta}$, or $h_{\alpha\beta}$ (or H_P, $N_{\alpha\beta}$, $m_{\alpha\beta}$, which are defined later).

6. Asymptotic formulae for the pressure structure function

We first discuss the asymptotic formulae for $D_P(r)$ for the inertial and dissipation ranges. At sufficiently large separations in the energy-containing range, $D_P(r) \approx D_P(\infty)$, which is twice the pressure variance that is given in (17) and (18).

To obtain $D_P(r)$ in the dissipation range, we use a Taylor series expansion of $Q(r)$. Substituting this expansion in (5) gives, as $r \rightarrow 0$,

$$D_P(r) = \tfrac{1}{3}\chi r^2 - \tfrac{1}{60}Q(0)\,r^4 + \dots. \tag{38}$$

Thus, $D_P(r)$ is quadratic at the origin, as is required by a Taylor series expansion of the pressure. Using (28) in (12) and the result in (10), or from (14), we have

$$\frac{Q(0)}{60} = d_{11} + \tfrac{1}{3}d_{\lambda\lambda} - 3d_{1\gamma} = d_{11}\,h_Q,$$

where the derivative moments $d_{\alpha\beta}$ are defined in (29), and d_{23} can be substituted for $d_{\lambda\lambda}/3$. Here, we define

$$h_Q \equiv 1 + \tfrac{1}{3}\Lambda_{\lambda\lambda} - 3\Lambda_{1\gamma}, \tag{39}$$

which is a universal constant at high Reynolds numbers. The first term on the right-hand side (38) is the viscous-range asymptotic formula for $r \rightarrow 0$, but both terms are important in the study of pressure-gradient correlations.

We now consider the inertial range. Substituting (32) in (13) gives

$$D_P(r) = H_P C_{11} \epsilon^{4/3} r^q = H_P D_{1111}(r). \tag{40a, b}$$

Equation (40*b*) follows from (40*a*) by use of (31), and H_P is a universal constant defined by

$$H_P \equiv \left[\frac{q}{3}(2+q) + \tfrac{8}{3}H_{\lambda\lambda} - 4(2+q)H_{1\gamma}\right] \Big/ [q(2-q)] \tag{41a}$$

$$\approx \frac{5}{3}\left(1-\frac{2\mu}{5}\right) + 3\left(1-\frac{\mu}{6}\right)H_{\lambda\lambda} - 15\left(1-\frac{7\mu}{30}\right)H_{1\gamma}. \tag{41b}$$

The approximation (41*b*) is (41*a*) given to the lowest order in μ. We also obtained the result (40*a*) by using (10), (12), and (32) to determine the inertial-range formula for $Q(r)$, and substituting it in (5). This verifies (40*a*). If we neglect intermittency effects (take $\mu = 0$), then C_{11} is a universal constant rather than having macrostructure dependence, and we define a new universal constant $C_P \equiv H_P C_{11}$; we obtain the simpler results

$$D_P(r) \approx \tfrac{5}{3}D_{1111}(r) + 3D_{\lambda\lambda\lambda\lambda}(r) - 15D_{11\gamma\gamma}(r) \tag{42a}$$

$$\approx C_P \epsilon^{4/3} r^{4/3}. \tag{42b}$$

Since $D_P(r) > 0$, (40*a*, *b*) give the bound $H_P > 0$ on the relative values of the structure-function components. This bound does not derive from kinematics alone; it results from use of the Navier–Stokes equation.

The inner scale ℓ_P of the pressure structure function is defined by equating (40*a*) and the first term of (38) at $r = \ell_P$. We obtain

$$\ell_P \equiv (3H_P C_{11} \epsilon^{4/3}/\chi)^{2-q}. \tag{43}$$

The relationship between the structure function and the spatial spectrum from data along a line is (Tatarskii 1971)

$$\Psi_P(k_1) = \frac{1}{\pi k_1^2}\int_0^\infty \mathrm{d}r \cos(k_1 r)\, D_P^{(2)}(r). \tag{44}$$

We have chosen the relationship given by Tatarskii (1971) that converges for $1 < q < 2$. Substituting (40*a*) in (44) gives the inertial-range formula

$$\Psi_P(k_1) = \pi^{-1}\Gamma(q+1)\cos[\tfrac{1}{2}\pi(q-1)]\, H_P C_{11} \epsilon^{4/3} k_1^{-q-1}. \tag{45}$$

For $q = 4/3$ and $C_P \equiv H_P C_{11}$, we have

$$\Psi_P(k_1) \approx 0.328 C_P \epsilon^{4/3} k_1^{-7/3}. \tag{46}$$

The inertial range of the three-dimensional wavenumber spectrum equals $-k\Psi_P^{(1)}(k)$, which is (45) multiplied by $(q+1)$ and with k_1 replaced by k; in the case of (46), $(q+1)$ becomes 7/3.

7. Asymptotic formulae for the mean-squared pressure gradient and the pressure-gradient correlation

We reduce the expression (19) for the mean-squared pressure gradient χ so that empirical knowledge combined with measurements of u_1 alone can give χ, and we further reduce it so that estimates of Reynolds number and energy dissipation rate suffice for estimation of χ in cases of large Reynolds numbers.

We define

$$H_\chi \equiv \int_0^\infty r^{-3}[D_{1111}(r)+D_{\lambda\lambda\lambda\lambda}(r)-6D_{11\gamma\gamma}(r)]\,\mathrm{d}r \Big/ \int_0^\infty r^{-3}D_{1111}(r)\,\mathrm{d}r. \quad (47)$$

Empirical determination of H_χ as a function of the Reynolds number allows χ to be obtained from measurements of u_1 alone because

$$\chi = 4H_\chi \int_0^\infty r^{-3}D_{1111}(r)\,\mathrm{d}r. \quad (48)$$

Since $D_{1111}(r) > 0$ and $\chi > 0$, from (48) or (19) we have the bound $H_\chi > 0$ on the relative values of the integrals of the structure function components in (47).

We now consider the case of low Reynolds numbers. Taylor's scale λ_T of the velocity correlation is given from

$$\lambda_T^2 \equiv \sigma_{11}/d_1. \quad (49)$$

Batchelor (1951) similarly defined the pressure lengthscale λ_P from

$$\lambda_P^2 \equiv \sigma_{11}^2/\langle(\partial_1 P)^2\rangle = 3\sigma_{11}^2/\chi. \quad (50)$$

For low Reynolds numbers, data for $D_{1111}(r)$ are shown in figure 1 in Batchelor (1951). Using these data in (48) and substituting the definitions (49) and (50), we obtain

$$H_\chi = 0.18(\lambda_T/\lambda_P)^2, \quad (51)$$

where the coefficient 0.18 in (51) is negligibly different from that obtained from the assumption of joint Gaussian velocities (Hill 1994). Values of λ_P/λ_T have been measured in dispersion experiments for low Reynolds numbers. As summarized by Monin & Yaglom (1975), these values of λ_P/λ_T scatter from 0.4 to 1.0. Numerical simulation of the Navier–Stokes equation would be more effective for obtaining H_χ for low to moderate Reynolds numbers. Numerical simulation by Schumann & Patterson (1978) gave the value $\lambda_P/\lambda_T \approx 2^{-1/2}$ for very low Reynolds numbers; this is the same as the value obtained from the assumption of joint Gaussian velocities by Uberoi (1954) and Hill (1994), and it is slightly smaller than the value given by Batchelor (1951). Therefore, for very low Reynolds numbers, (51) gives $H_\chi \approx 0.36$.

We now turn to the case of large Reynolds numbers such that a substantial inertial range exists. Using (35), we have

$$\int_0^\infty \mathrm{d}r\, r^{-3} D_{\alpha\alpha\beta\beta}(r) = C_{\alpha\beta}\,\epsilon^{4/3}\,\ell_{\alpha\beta}^{q-2}\,N_{\alpha\beta}, \quad (52)$$

where

$$N_{\alpha\beta} \equiv \int_0^\infty \mathrm{d}r_{\alpha\beta}\, r_{\alpha\beta}^{-3}\,\tilde{D}_{\alpha\alpha\beta\beta}(r_{\alpha\beta}). \quad (53)$$

For large enough Reynolds number, the energy-containing range makes a negligible contribution to (53). Let the viscous-range contribution to (53) be the integral over $r_{\alpha\beta}$ less than some chosen number. This number is taken to be independent of α and β and is significantly less than unity. Let the inertial-range contribution to (53) be the integral for $r_{\alpha\beta}$ greater than some chosen number that is independent of α and β and is significantly greater than unity. From (53) and (35), we see that these contributions to $N_{\alpha\beta}$ from both the inertial and viscous ranges are independent of α and β because $\tilde{D}_{\alpha\alpha\beta\beta}(r_{\alpha\beta}) = r_{\alpha\beta}^4$ in the viscous range and $\tilde{D}_{\alpha\alpha\beta\beta}(r_{\alpha\beta}) = r_{\alpha\beta}^q$ in the inertial range. Thus, for various choices of α and β, the $N_{\alpha\beta}$ differ only because of differences in the transition between inertial and viscous ranges of the various $\tilde{D}_{\alpha\alpha\beta\beta}(r_{\alpha\beta})$. The $N_{\alpha\beta}$ are dimensionless numbers of order unity. For example, we obtain $N_{\alpha\beta} = 3/2$ from the *ad hoc* formula

$$\tilde{D}_{\alpha\alpha\beta\beta}(r_{\alpha\beta}) = r_{\alpha\beta}^4(1+r_{\alpha\beta}^2)^{(q-4)/2}. \tag{54}$$

This formula (54) interpolates between the asymptotic formulae in the viscous and inertial ranges where $\tilde{D}_{\alpha\alpha\beta\beta}(r_{\alpha\beta})$ equals $r_{\alpha\beta}^4$ and $r_{\alpha\beta}^q$, respectively. Since the $N_{\alpha\beta}$ depend on the shape of the transition between inertial and viscous ranges, they might depend on the Reynolds number. We now define

$$m_{\alpha\beta} \equiv N_{\alpha\beta}/N_{11}. \tag{55}$$

The $m_{\alpha\beta}$ depend on the relative shapes of the viscous-to-inertial-range transition, and the $m_{\alpha\beta}$ are therefore presumed to be independent of the Reynolds number, or very nearly so, and close to unity.

Substituting (52) and (55) in (47) gives

$$H_\chi = 1 + m_{\lambda\lambda} h_{\lambda\lambda} - 6m_{1\gamma} h_{1\gamma}, \tag{56}$$

where the universal constants $h_{\lambda\lambda}$ and $h_{1\gamma}$ are defined in (37). Thus, H_χ is a universal constant for large Reynolds numbers, and H_χ need be determined for only one sufficiently large Reynolds number. By relating H_χ to $h_{\lambda\lambda}$ and $h_{1\gamma}$ in (56), we show that H_χ depends on the ratios of derivative moments $A_{\alpha\beta}$ and ratios of inertial-range levels $H_{\alpha\beta}$ as in (37), with little effect from the details of the transition between inertial and viscous ranges as parameterized by $m_{\alpha\beta}$.

Substituting (52) in (48) gives, for large Reynolds numbers,

$$\chi = 4N_{11} H_\chi C_{11} \epsilon^{4/3} \ell_{11}^{q-2} \tag{57a}$$

$$= 4N_{11} H_\chi (C_{11}\epsilon^{4/3})^{2/(4-q)} (\epsilon/15\nu)^{2(2-q)/(4-q)} K_{11}^{(2-q)/(4-q)}, \tag{57b}$$

where $\epsilon = 15\nu d_1$, ν is kinematic viscosity, and (57*b*) is obtained from (57*a*) using (30) and (34). Experiments can establish K_{11} and N_{11} as functions of the Reynolds number. Wyngaard & Tennekes (1970) gave this information for K_{11}. In the absence of measurements, N_{11} can be estimated to be about 3/2, as we obtained from (53) and (54). A measurement of the inertial range of $D_{1111}(r)$ gives a value of $C_{11}\epsilon^{4/3}$ for use in (57*b*); ϵ can be estimated from this inertial range of $D_{1111}(r)$, or ϵ can be estimated by other means, including from the inertial range of the second-order structure function. The estimate of ϵ and a measurement of σ_{11} give the Reynolds number, from which K_{11} and N_{11} can be obtained. If a measurement of H_χ is performed at a high Reynolds number, then the required empirical knowledge will be available such that for all high-Reynolds-number turbulence, (57*b*) gives an estimate of χ from measurements no more complicated than the inertial range of $D_{1111}(r)$. The explicit dependence of (57*b*) on ϵ and ν is approximately $\nu^{-1/2}\epsilon^{3/2}$, but there is also implicit dependence on ϵ and ν from the Reynolds-number dependencies of the other quantities.

Substituting (57*a*) in (43), we obtain for the pressure inner scale

$$\ell_P = \left(\frac{3H_P}{4N_{11} H_\chi}\right)^{1/(2-q)} \ell_{11}. \tag{58}$$

If H_P differs greatly from $2H_\chi$ (taking $N_{11} \approx 3/2$), then ℓ_P differs greatly from ℓ_{11}. Then part of the inertial range of $D_P(r)$ coincides with the viscous range of $D_{1111}(r)$, or vice versa. This strange possibility contradicts the notion that pressure differences and velocity differences are interdependent. This strange possibility suggests that ℓ_P and ℓ_{11} are of the same order of magnitude and, therefore, that $3H_P/4N_{11}H_\chi$ is of the order of unity. If so, a measurement of H_P gives a rough estimate of H_χ. Therefore, for the purpose of roughly estimating the mean-squared pressure gradient, we can replace the factor $4N_{11}H_\chi$ in (57*b*) with $3H_P$. Because the estimate is rough, we further simplify by using $\mu = 0$; we then obtain

$$\chi \approx 3H_P(C_{11}\epsilon^{4/3})^{3/4}(\epsilon/15\nu)^{1/2}K_{11}^{1/4}, \tag{59a}$$

wherein the required new empirical knowledge is H_P, which requires only inertial-range measurements. On the other hand, to make use of (57*b*) requires empirically determined values of N_{11} and H_χ; obtaining these values requires both inertial- and dissipation-range measurements. We can express (59*a*) in terms of the inertial-range flatness factor $F \equiv C_{11}/C_1^2$, where C_1 is the Kolmogorov constant of the second-order structure function of the longitudinal velocity component. We use the value $C_1 = 2$, as recommended by Yaglom (1981). We then have from (59*a*) that

$$\chi \approx 2.2H_P\epsilon^{3/2}\nu^{-1/2}F^{3/4}K_{11}^{1/4}. \tag{59b}$$

For Reynolds number varying from low values observed in the laboratory to high values observed in the atmospheric surface layer, Wyngaard & Tennekes (1970) showed that K_{11} varies from about 4 to 40 and Van Atta & Chen (1970) showed that F varies from about 3 to 10. These observations imply that the factor $F^{3/4}K_{11}^{1/3}$ in (59*b*) increases by a factor of about 4 over the same range of Reynolds numbers. Estimates of χ on the basis of the assumption of joint Gaussian velocities or velocity derivatives (Yaglom 1949; Batchelor 1951; Hill 1994) do not have this Reynolds-number dependence, and these estimates are too small by the above-mentioned factor of about 4 for the case of the high Reynolds numbers observed in the atmospheric surface layer.

We now establish asymptotic formulae and lengthscales relevant to the pressure-gradient correlation $A_{ij}(\boldsymbol{r})$. For the viscous range, we differentiate (38), as required in (22*a*, *b*), to obtain

$$A_{\alpha\alpha}(r) = \tfrac{1}{3}\chi - n_\alpha d_{11}h_Q r^2 + \cdots = \tfrac{1}{3}\chi(1 - r^2/2\lambda_\alpha^2 + \cdots), \tag{60a, b}$$

where $n_1 = 6$, $n_2 = n_3 = 2$, and

$$\lambda_\alpha^2 \equiv \chi/6n_\alpha d_{11}h_Q. \tag{61}$$

The length λ_α is the spacing at which $A_{\alpha\alpha}(r) \approx A_{\alpha\alpha}(0)/2$; λ_1 is smaller than λ_2 or λ_3.

For low Reynolds numbers, we use (30), (49), and (50) in (61) to obtain

$$\lambda_\alpha = \lambda_T/[2(\lambda_P/\lambda_T)^2 n_\alpha h_Q K_{11}]^{1/2} \approx \lambda_T/(3n_\alpha h_Q)^{1/2}. \tag{62a, b}$$

To obtain (62*b*) from (62*a*), we use $\lambda_P/\lambda_T \approx 2^{-1/2}$ and $K_{11} \approx 3$, as is appropriate for low Reynolds numbers.

For large Reynolds numbers, we use (57*a*) in (61) to obtain

$$\lambda_\alpha = (2N_{11}H_\chi/3n_\alpha h_Q)^{1/2}\ell_{11}. \tag{63}$$

Using (58), we can also relate λ_α to ℓ_P.

For the inertial range, we substitute (40*a*, *b*) into (22*a*, *b*) to obtain

$$A_{\alpha\alpha}(r) = n'_\alpha H_P r^{-2}D_{1111}(r) = \tfrac{1}{3}\chi[n'_\alpha(\ell_P/r)^{2-q}], \tag{64a, b}$$

where $n'_2 = n'_3 = q/2$ and $n'_1 = (q-1)n'_2$. The quantity in square brackets in (64*b*) expresses $A_{\alpha\alpha}(r)$ as a fraction of $A_{\alpha\alpha}(0) = \chi/3$. For the inertial range, we see that $A_{\alpha\alpha}(r)$ is positive and decreases approximately as $r^{-2/3}$, and that $A_{11}(r)$ is about one-third of $A_{\lambda\lambda}(r)$.

Consider the transition range between the initial decrease (60*b*) at small r, where

$A_{11}(r)$ decreases more rapidly than $A_{\lambda\lambda}(r)$, and the inertial range (64*a*, *b*), where $A_{11}(r)$ and $A_{\lambda\lambda}(r)$ are both positive and decreasing. Whether $A_{\lambda\lambda}(r)$ and $A_{11}(r)$ are monotonically decreasing or have local minima or even negative values depends on the narrowness of the transition between the viscous and inertial ranges of $D_P(r)$; $A_{11}(r)$ is more sensitive to the details of this transition than is $A_{\lambda\lambda}(r)$. The narrowness of the transition is parameterized by ℓ_P/λ_1. From our assumption that ℓ_P and ℓ_{11} are of the same order of magnitude, we see from (58) and (63) that the parameter $(H_\chi/6h_Q)^{1/2}$ is important to determining ℓ_P/λ_1. It seems likely that $D_P^{(1)}(r)$ is monotonically increasing in the transition between its increase as r^1 in the viscous range and as r^{q-1} in the inertial range. If so, $D_P^{(1)}(r)$ is positive and monotonically increasing from $r = 0$ to r at the large-scale end of the inertial range and, from (22*a*), $A_{\lambda\lambda}(r)$ is positive and monotonically decreasing at these r. Then $A_{\lambda\lambda}^{(1)}(r) < 0$ and (22*a*, *b*) require $A_{11}(r) < A_{\lambda\lambda}(r)$. Furthermore, if the transition is sufficiently narrow (we roughly estimate $\ell_P/\lambda_1 < 2.7$), then $A_{11}(r)$ can have negative values in the transition. From (60*b*), this must occur at $r > \lambda_1$, and $A_{11}(r)$ must cross zero again to become positive in the inertial range, as required by (64*a*, *b*). Obukhov & Yaglom (1951) obtained this same behaviour of locally negative $A_{11}(r)$ on the basis of the assumption of joint Gaussian velocity derivatives; that is, the joint Gaussian assumption produces a sufficiently narrow transition.

8. Sensitivity to departures from isotropy and incompressibility

George, Beuther & Arndt (1984) measured the pressure spectrum in the mixing layer of an axisymmetric jet and compared it with theory. They showed that their spectrum is caused by three terms that they called (i) 2nd-moment turbulent–shear interaction, (ii) 3rd-moment turbulence–shear interaction, and (iii) turbulence–turbulence interaction. Our $D_P(r)$ is caused by only this last (iii) interaction. Within the inertial range observed by George *et al.* (1984), we deduct their error spectrum and their spectrum of the combined turbulence–shear interactions from their measured pressure spectrum. We thereby obtain that the portion of their measured spectrum attributable to the turbulence–turbulence interaction is a factor of about 1.5 greater than their theoretical turbulence–turbulence interaction spectrum. That is, judging by comparison of the theoretical curves and measured spectra in their figure 15, we have that

$$D_P(r) \approx 1.5\ [D_{11}(r)]^2, \text{ where } D_{11}(r) \equiv \langle (u_1 - u_1')^2 \rangle.$$

The second-order structure function appears because George *et al.* (1984) used Batchelor's (1951) formula for the turbulence–turbulence interaction.

Antonia *et al.* (1982) obtained the inertial-range flatness factor $D_{1111}(r)/[D_{11}(r)]^2 \approx 4.5$; this was obtained from an axisymmetric jet having nearly the same Reynolds number (based on nozzle diameter *d* and exit velocity) as in the experiment by George *et al.* (1984), but Antonia *et al.* used the downstream position 50*d*, whereas George *et al.* used 1.5*d* and 3.0*d*. Therefore, we assume that the flatness factor was about 4.5 in the experiment by George *et al.* Further support for a flatness factor of 4.5 is obtained by calculating the Reynolds number (based on Taylor's scale) for the data by George *et al.* (1984), thereby obtaining the flatness from figure 12 by Antonia *et al.* (1982).

Using this estimate of the flatness factor and the observation from the George *et al.* data that $D_P(r) \approx 1.5\ [D_{11}(r)]^2$, we obtain $D_P(r) \approx D_{1111}(r)/3$ in the inertial range, implying from (40*b*) that $H_P \approx 1/3$. Further support for $H_P \approx 1/3$ can be obtained from the numerical simulations by Fung *et al.* (1992) and Métais & Lesieur (1992). They calculated pressure spectra that show inertial ranges and they stated their values of the Kolmogorov constant. However, they did not give a value of the inertial-range flatness factor; consistent with their moderate Reynolds numbers, we assume that this factor is 3. In this case, we obtain $H_P = 0.327$ and 0.338 from the simulations by Fung *et al.* (1992) and Métais & Lesieur (1992), respectively. If $H_P \approx 1/3$, then the three terms in (42*a*) cancel each other to produce a $D_P(r)$ that is about five times smaller than the first term in (42*a*). From measured values of $H_{\lambda\lambda}$ and $H_{1\gamma}$ given in §9, we find that if $H_P \approx 1/3$ then $D_P(r)$ is respectively about 14 and 19 times smaller than the second and third terms in (42*a*). We introduce the notion of a flow being sufficiently isotropic (or locally isotropic) for $D_P(r)$ to be calculated from $D_{ijkl}(\boldsymbol{r})$ to within some chosen accuracy. The inertial-range formula (42*a*) gives a stringent requirement on sufficiency of local isotropy because $D_P(r)$ is so much smaller than the individual terms in (42*a*). The formulae for the viscous range of $D_P(r)$ and the mean-square pressure gradient are also sensitive to the accuracy of isotropy. Hill (1993) shows that the theory of 1948–1951 has this same sensitivity to local isotropy of data for the same reasons.

We note that even if a flow is sufficiently isotropic, or sufficiently locally isotropic at some r, velocity data can be insufficiently isotropic because of imperfections in the measurement process (Karyakin, Kuznetsov & Praskovskii 1991). For instance, when the energy-containing range is anisotropic, use of Taylor's hypothesis can result in measured local anisotropy even if the flow has accurate local isotropy (Hill 1995). Care must therefore be exercised when using Taylor's hypothesis to obtain fourth-order velocity structure functions for use in (40*a*, *b*). Recent measurements of the approach to local isotropy are described by Saddoughi & Veeravalli (1994).

On the basis of our discussion of (58), we obtained that $4N_{11} H_\chi$ is of the order of $3H_P$ for large Reynolds numbers. Also, we obtained $N_{11} \approx 3/2$. If our estimate that $H_P \approx 1/3$ is correct, then we have that H_χ is about 0.2 for large Reynolds numbers. If so, then the integrals in the numerator of (47) cancel to give a value that is about 20% of the first term in the numerator of (47) and is perhaps a yet smaller percentage of the largest term, whichever term that may be. (For instance, according to the joint Gaussian assumption, the third term in the numerator of (47) is almost -3 times the first term (cf. Hill 1994).) Hence, we expect that H_χ will be difficult to obtain using (47) and measurements of $D_{ijkl}(\boldsymbol{r})$, and that measurements must accurately obey local isotropy for use in (47).

Incompressibility is a very important assumption in our derivations; it produces great simplification relative to the case of compressible fluid flow. To derive the pressure structure function and pressure spectrum for the case of compressibility requires that we use the hydrodynamics equations for the compressible case. The derivation would be much more complicated than that presented in this paper. However, some compressibility effects can be expressed in terms of the non-vanishing of the quantity $M_{ijkl}(\boldsymbol{r})_{|ijkl}$. Hill (1993) showed that neglecting $M_{ijkl}(\boldsymbol{r})_{|ijkl}$ requires increasingly stringent accuracy of the incompressibility condition with increasing Reynolds number and with decreasing r within the inertial range.

9. Experiment

We examined data from the atmospheric surface layer to determine the values of $H_{\lambda\lambda}$ and $H_{1\gamma}$. The data were obtained at a flat agricultural site having nearly uniform surface characteristics over a fetch of more than 1 km; the site and experiment are described by Oncley (1992). To determine $H_{\lambda\lambda}$ and $H_{1\gamma}$, we need simultaneous measurement of all three components of velocity. At least two components must be measured at a single spatial location. We obtained all three components at a single

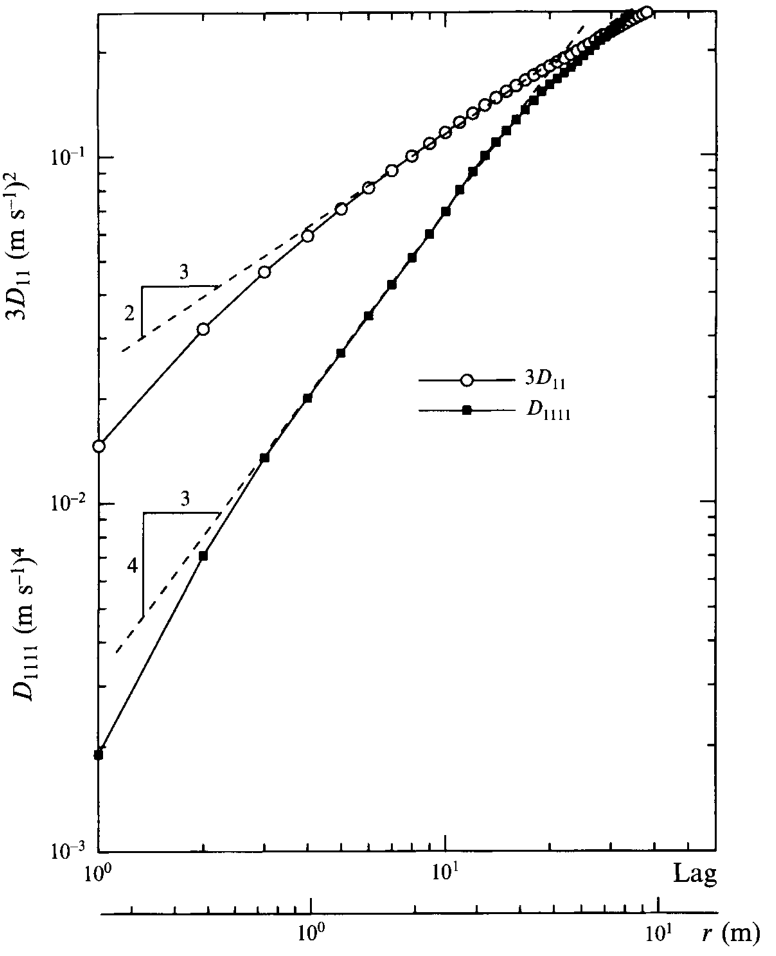

FIGURE 1. $D_{11}(r)$ and $D_{1111}(r)$ as functions of lag and r, as well as 4/3 and 2/3 power laws. $D_{11}(r)$ was multiplied by 3 to avoid crossover of all curves within the inertial range.

location 7 m above ground using a three-axis sonic anemometer having 0.2 m gaps between transducers. This anemometer is described by Zhang *et al.* (1986). The data were sampled at 20 Hz. Each sample is an average over 20 sonic pulses, so Taylor's hypothesis implies a spatial averaging along the streamwise direction. Data from three stationary 60 min. periods were analysed. The stabilities for these three periods were nearly neutral ($z/L_O = 0.001$), slightly stable ($z/L_O = 0.04$), and moderately unstable ($z/L_O = -0.09$), where z is height and L_O is Obukhov length. Of the three periods, the unstable period displayed inertial-subrange behaviour over the broadest range of lengthscales, and we present data only from this run, although the results from all three periods are similar.

For the unstable data run, the Obukhov length was -80 m and the wind speed at 7 m was $U = 4.9$ m s^{-1}. The velocity covariance tensor in units of (m s^{-1})^2 was $\sigma_{11} = 0.64$, $\sigma_{12} = 0.054$, $\sigma_{13} = -0.074$, $\sigma_{22} = 0.59$, $\sigma_{23} = 0.034$, $\sigma_{33} = 0.12$, where the indices 1, 2 and 3 respectively denote the streamwise, horizontal cross-stream, and vertical components of velocity. Clearly, the turbulence was very anisotropic at large scales.

Figure 1 shows $D_{1111}(r)$ and $D_{11}(r)$. Lag on the abscissa is the number of consecutive samples producing spacing $r = \text{lag} \times (U/20 \text{ Hz})$. The 4/3 and 2/3 approximate power

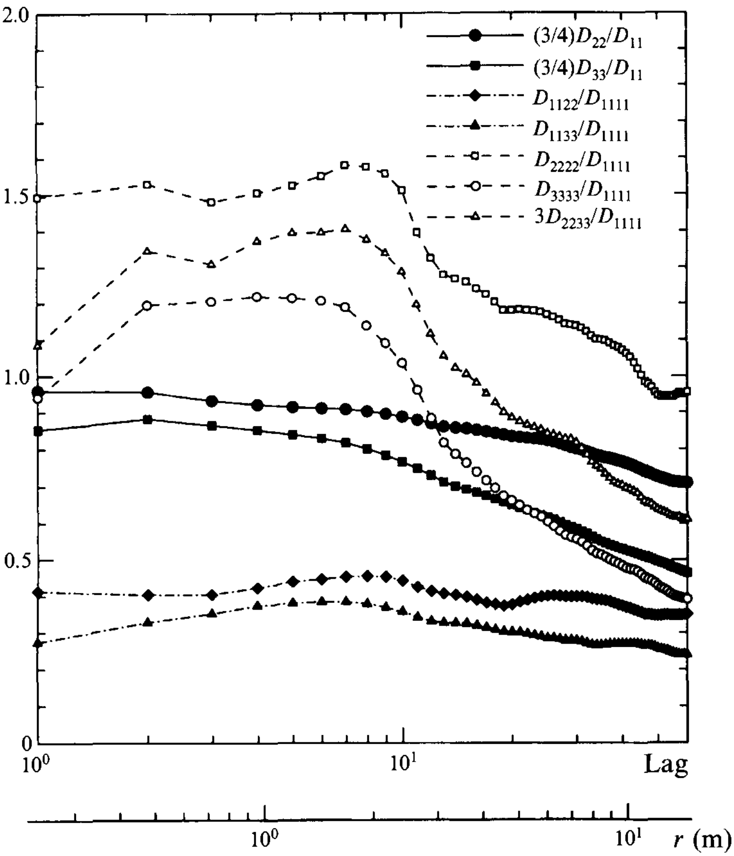

FIGURE 2. Ratios of structure function components to the streamwise components $D_{11}(r)$ and $D_{1111}(r)$.

laws are evident from about $r = 1$ m to half the height above ground for $D_{1111}(r)$ and to the height above ground for $D_{11}(r)$. Of course, the $(4/3)+(2\,\mu/9)$ power law for $D_{1111}(r)$ was established in other experiments (e.g. Van Atta & Chen 1970; Anselmet *et al.* 1984). For $r < 1$ m, the space-averaging effect of the sonic anemometer causes the curves to depart from the power laws. We considered correcting the anemometer for this spatial averaging, but the anemometer averages in a complex way and the correction would not alter our results.

The data were selected to maximize the accuracy of Taylor's hypothesis, which was used to convert time to space. Indeed, the quantity σ_{ij}/U^2, which governs the correction to Taylor's hypothesis, is significantly smaller for our data than for the moderately unstable atmospheric surface-layer values given by Wyngaard & Clifford (1977). Nevertheless, we were concerned that the observed small-scale anisotropy might be caused by errors in Taylor's hypothesis. Therefore, we corrected our statistics at small spatial scales (lag $\leqslant$ 12) using the formulae by Hill (1995).

In figure 2, we show ratios of structure-function components as corrected for errors in Taylor's hypothesis. The curves of $(3/4)\,D_{22}(r)/D_{11}(r)$ and $(3/4)\,D_{33}(r)/D_{11}(r)$ should approach unity as local isotropy is approached with decreasing r. In the apparent power-law range, $1 < r < 7$ m, these curves are tending toward unity, but show considerable anisotropy. The ratios $D_{12}(r)/D_{11}(r)$, $D_{23}(r)/D_{11}(r)$, and $D_{13}(r)/D_{11}(r)$ should be zero in locally isotropic turbulence. Although not shown in the figure, the former two ratios are between 0.07 and 0.02 in the observed power-law range and the last ratio increases toward zero from -0.2 at $r = 7$ m to -0.06 at $r = 1$ m. Because $D_{13}(r)$ is related to the momentum flux, it is expected to be negative and increasing toward zero with decreasing r. Also shown in figure 2 are the ratios $D_{2222}(r)/D_{1111}(r)$, $D_{3333}(r)/D_{1111}(r)$, and $3D_{2233}(r)/D_{1111}(r)$. These should all have the same value $H_{\lambda\lambda}$ in locally isotropic turbulence. Figure 2 shows that they differ and are increasing over the large-r half of the observed power-law range. These ratios are nearly constant in the range of 4 to 7 lags ($1 < r < 2$ m), where their values range from 1.2 to 1.6. Likewise, the ratios $D_{1122}(r)/D_{1111}(r)$ and $D_{1133}(r)/D_{1111}(r)$ should become the universal constant $H_{1\gamma}$ as local isotropy is approached. However, figure 2 shows that they differ and vary somewhat in the observed power-law range. Their values are within the range 0.37 to 0.45.

Since $D_{22}(r)/D_{11}(r)$ is closer to its isotropic inertial-range value than is $D_{33}(r)/D_{11}(r)$, presumably because the vertical component of velocity is related to momentum flux, we take our estimates of $H_{\lambda\lambda}$ and $H_{1\gamma}$ from $D_{2222}(r)$ and $D_{1122}(r)$, respectively. Using lag = 5 and figure 2, we then have that H_{22} is about 1.5, and obtain that H_{12} is about 0.43. The statistical uncertainty is negligible compared with the uncertainty caused by anisotropy because our 60 min. averages have excellent statistical reliability. The correction for errors in Taylor's hypothesis are small and affect the ratios D_{2222}/D_{1111}, D_{3333}/D_{1111}, and D_{2233}/D_{1111} more than the other ratios. For instance, at lag = 5, D_{2222}/D_{1111} is increased 2% by the correction. Substituting $H_{\lambda\lambda} = 1.5$ and $H_{1\gamma} = 0.43$ in (41 *a*), we obtain $H_P = -0.26$, which is impossible because $H_P > 0$. The value $H_P = -0.26$ is only 4% of the largest (last) term in (41 *a*). Clearly, we have not determined these constants to the accuracy required to determine H_P. Indeed, if we use the data in figure 2 to calculate H_P in all five ways permitted by isotropy, we obtain inconsistent values.

For the case of the assumption of joint Gaussian velocities or velocity derivatives, which was used in the theory of 1948–1951, Hill (1993) gave five formulae relating $D_P(r)$ to components of the second-order velocity structure function. Given local isotropy, these five formulae must yield the same value of $D_P(r)$. We evaluated these five formulae using our data. We obtain inconsistent values of $D_P(r)$ including negative values. Thus, the theory of 1948–1951 and our theory are both sensitive to local anisotropy, as was demonstrated by Hill (1993).

10. Summary and conclusions

Starting from the concept that statistics of products of pressure differences and of such products multiplied by velocity differences should be related to statistics of velocity differences, we relate $D_P(r)$ to $D_{ijkl}(\boldsymbol{r})$. The pressure spectrum and correlation, as well as pressure-gradient correlations and mean-squared pressure gradient, therefore are also related to $D_{ijkl}(\boldsymbol{r})$. Because $D_P(r) > 0$, we have the bounds $H_P > 0$ and $H_\chi > 0$ on the relative magnitudes of the components of $D_{ijkl}(\boldsymbol{r})$ in the inertial and viscous ranges. Our results open a new door to experimentation on pressure statistics in high-Reynolds-number turbulence. We derive $D_P(r)$ without using the assumption of joint Gaussian velocities, nor do we use any replacement approximation. This makes our method a natural beginning point for pressure-related statistics in compressible fluids and in anisotropic turbulence, such as atmospheric turbulence. The formulation is compared by Hill (1994) with the formulation obtained using the assumption of joint Gaussian velocities.

By relating $D_P(r)$ to $D_{ijkl}(\boldsymbol{r})$, we develop a useful relationship for locally isotropic and locally homogeneous turbulence, even if the energy-containing range is anisotropic and inhomogeneous. One component of $D_{ijkl}(\boldsymbol{r})$ has been the subject of extensive experimental and theoretical investigation. As a result, for the inertial range we obtain $D_P(r) \propto r^{4/3}$, as derived by Obukhov (1949) and Batchelor (1951), or slightly different from $r^{4/3}$, as indicated in (40*a*, *b*). The inertial-range proportionality factor H_P given in (41*a*, *b*) or (42*a*) has three terms involving the level of three components of the structure function. The three structure-function components should be obtained from measurements or from numerical simulation of the Navier–Stokes equation.

Using the data by George *et al.* (1984), we obtain the rough estimate that $H_P \approx 1/3$, which is also the value obtained on the basis of the joint Gaussian assumption (Hill 1994). If this estimate is accurate, then the discrepancy between the theory of 1948–1951 and the experiment as shown by George *et al.* (1984) is attributable to the fact that $D_{1111}(r)$ deviates significantly from its joint Gaussian approximation. Numerical simulations by Métais & Lesieur (1992) and Fung *et al.* (1992) also suggest that $H_P \approx 1/3$.

For the case of very low Reynolds numbers, we have estimated that $H_\chi \approx 0.36$ on the basis of data for $D_{1111}(r)$ shown by Batchelor (1951) and numerical simulation by Schumann & Patterson (1978). Numerical simulation can give more reliable values of H_χ. For the case of very large Reynolds numbers, we estimate in §8 that $H_\chi \approx 0.2$.

When possible, it is important to delineate the applicability of assumptions used in any theory. In §§8 and 9, we delineate the sensitivity of our asymptotic inertial-range formulae to local anisotropy. Such anisotropy is present to some degree in any experimental data. We show that H_P is sensitive to the accuracy of local isotropy in any given data, and (41*b*) expresses the sensitivity of H_P to the inertial-range exponent of the structure function components. As shown by Hill (1993), the joint Gaussian assumption does not mitigate this sensitivity to local anisotropy of data. Thus, the theory of 1948–1951 has sensitivity to anisotropy of data that is commensurate with that of the present theory. Hill (1993) noted that use of the theory of 1948–1951 requires measurement of at least one transverse velocity component, as well as the longitudinal component, such that sufficiency of isotropy can be demonstrated. Although our theory is challenging to evaluate experimentally, so is the theory of 1948–1951.

In the introduction, we criticized relating $D_P(r)$ to $R_{ijkl}(\boldsymbol{r})$ because the subtraction of very large values of $R_{ijkl}(\boldsymbol{r})$ produces relatively small quantities needed for pressure statistics. Above, we state that pressure statistics are sensitive to cancellation of terms containing components of $D_{ijkl}(\boldsymbol{r})$. It is important to emphasize that these sensitivities are not commensurate. Taken together, (6*a*) and (6*c*) relate components of $D_{ijkl}(\boldsymbol{r})$ to those of $R_{ijkl}(\boldsymbol{r})$. Let L be a scale characteristic of the energy-containing range. In the inertial range, components of $R_{ijkl}(\boldsymbol{r})$ are larger than those of $D_{ijkl}(\boldsymbol{r})$ by factors of order $(L/r)^{4/3}$. Thus, if we use (6*a*) and (6*c*) to introduce components of $R_{ijkl}(\boldsymbol{r})$ into our equations, e.g. (42*a*), then we have amplified the sensitivity to cancellation of terms by a factor of $(L/r)^{4/3}$. For increasing Reynolds number, the factor $(L/r)^{4/3}$ can increase without bound.

We present the first measured values of the constants $H_{\lambda\lambda}$ and $H_{1\gamma}$, obtaining $H_{\lambda\lambda} = 1.5$ and $H_{1\gamma} = 0.43$. However, the measurements are not sufficiently accurate to reliably obtain H_p. Determining H_P requires very accurate measurements of velocity for flows exhibiting greater local isotropy. Resolving smaller r would be essential for the atmospheric surface layer. More nearly isotropic turbulence at energy-containing range scales would be useful. To accurately obtain H_P, one must consider even as small a correction to Taylor's hypothesis as was obtained from our data. Our experimental results show that there should be caution in future experimental attempts. The other universal constants, H_χ and h_Q, might also be difficult to measure for the same reasons. Each of the sums and differences of structure-function components that appear in our formulae can be expressed as a single average; a simple example is, using (11),

$$D_{\lambda\lambda\lambda\lambda}(r) - 3D_{11\gamma\gamma}(r) = 3\langle[(u_2-u_2')^2-(u_1-u_1')^2](u_3-u_3')^2\rangle.$$

Although there is no mathematical difference between these two expressions, measurement techniques might be devised to exploit the expression on the right-hand side.

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Appendix

Here we derive (4*a*, *b*), (7), (8), (9) and (10) on the basis of local homogeneity. Local isotropy is not used. Although local homogeneity and local isotropy are often used for the case of large Reynolds numbers and inertial-range (or smaller) scales, any result derived on the basis of local homogeneity or local isotropy is also valid under assumptions of homogeneity or isotropy, respectively, independent of Reynolds number or spatial scale.

Consider any statistic containing at least one difference of a turbulence quantity, e.g. $(P-P')$ or (u_i-u_i'), or containing at least one derivative. Local homogeneity means that such a statistic is very rapidly varying with respect to $\boldsymbol{r}$ relative to its variation with respect to $\boldsymbol{X} \equiv (\boldsymbol{x}+\boldsymbol{x}')/2$, provided that r is sufficiently small. The operational calculus of local homogeneity is that derivatives with respect to x_i (and x_k') within the averaging operation can be commuted to outside the average such that they become derivatives with respect to r_i (and $-r_k$, as the case may be); also, derivatives with respect to r_i can be commuted to inside the average where they may be performed with respect to either x_i or $-x_i'$. A case in point is the following result that we need:

$$\begin{aligned} Q(\boldsymbol{r}) &\equiv \langle \partial_{ij}(u_i u_j)\, \partial_{kl}'(u_k' u_l')\rangle = \langle (u_i u_j)\, \partial_{kl}'(u_k' u_l')\rangle_{|ij} \\ &= \langle (u_i u_j)\, \partial_{ij}' \partial_{kl}'(u_k' u_l')\rangle = \langle (u_i u_j)\, \partial_{ij}'(u_k' u_l')\rangle_{|kl} \\ &= \langle \partial_{kl}(u_i u_j)\, \partial_{ij}'(u_k' u_l')\rangle = \langle \partial_{ij}(u_k u_l)\, \partial_{kl}'(u_i' u_j')\rangle. \end{aligned} \tag{A 1}$$

The last step in (A 1) is obtained by relabelling the indices. We suppress the dependence on $\boldsymbol{X}$ for a statistic that obeys local homogeneity, as does $Q(\boldsymbol{r})$.

We now obtain (4*a*) on the basis of local homogeneity. We note that

$$\partial_{ii}\partial_{kk}'(P-P')^2 = \partial_{ii}\partial_{kk}'(P^2-2PP'+P'^2) = -2\partial_{ii}P\partial_{kk}'P'; \tag{A 2}$$

thus,

$$\partial_{ii}\partial_{kk}'(P-P')^2 = -2\rho^2\,\partial_{ij}(u_i u_j)\,\partial_{kl}'(u_k' u_l'), \tag{A 3}$$

where (A 2) is an identity and (A 3) follows by substitution of (3) in (A 2). Averaging (A 3) and using local homogeneity, we obtain

$$\langle (P-P')^2\rangle_{|iikk} = -2\rho^2 Q(\boldsymbol{r}), \tag{A 4}$$

which is equivalent to (4 *a*). Thus, derivation of (4*a*) on the basis of local homogeneity is as easy and concise as using homogeneity.

Performing the divergence of (6*e*) and using local homogeneity, we have

$$B_{ijkl}(\boldsymbol{r})_{|i} = \langle -(\partial_i' u_i') u_j u_k u_l \rangle + \langle -(\partial_i u_i) u_j' u_k' u_l' \rangle, \tag{A 5}$$

where we chose to convert the divergence with respect to $\boldsymbol{r}$ into one with respect to $\boldsymbol{x}'$ inside the leftmost average, and with respect to $\boldsymbol{x}$ in the rightmost average. By incompressibility, $\partial_i u_i = 0$ and $\partial_i' u_i' = 0$, so (7) follows. Performing the fourth-order divergence of (6*d*), we see that each of the four terms has one divergence summed over its leftmost index. Therefore, each of the four terms in (6*d*) vanishes because of (7). Therefore, both (7) and (8) are valid on the basis of local homogeneity.

Performing the fourth-order divergence of (6*b*), we obtain

$$\begin{aligned} S_{ijkl}(\boldsymbol{r})_{|ijkl} &= \langle (u_i u_j - u_i' u_j')(u_k u_l - u_k' u_l') \rangle_{|ijkl} && \text{(A 6a)} \\ &= \langle -(u_k u_l - u_k' u_l') \partial_{kl}'(u_i' u_j') - (u_i u_j - u_i' u_j') \partial_{kl}'(u_k' u_l') \rangle_{|ij} && \text{(A 6b)} \\ &= -\langle \partial_{ij}(u_k u_l) \partial_{kl}'(u_i' u_j') + \partial_{ij}(u_i u_j) \partial_{kl}'(u_k' u_l') \rangle && \text{(A 6c)} \\ &= -2Q(\boldsymbol{r}), && \text{(A 7)} \end{aligned}$$

where obtaining (A 6*b*, *c*) from (A 6*a*) requires local homogeneity and the calculus of differentiation, and (A 7) follows from (A 6*c*) using (A 1). Performing the fourth-order divergence of (6*a*) and substituting (8) and (A 7) yields

$$D_{ijkl}(\boldsymbol{r})_{|ijkl} = -3S_{ijkl}(\boldsymbol{r})_{|ijkl} = 6Q(\boldsymbol{r}), \tag{A 8}$$

which proves (10) on the basis of local homogeneity.

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